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ON CERTAIN PROPERTIES OF SOLUTIONS OF SINGULARLY PERTURBED SYSTEMS IN A PARTICULAR CRITICAL CASE*

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Systems defined by differential equations with a small parameter at derivatives are considered. Methods of the theory of motion stability are used for showing that in the critical Liapunov case of several zero roots solution of the problem of stability for a complete system can be reduced to solving the problem for an approximate system of a lower order, taken as the simplified problem. Conditions under which the respective solutions of the complete and the simplified systems are close to each other in an infinite interval of time are presented. Gyroscopic systems containing gyroscopes with large proper moment of momentum are used as an example of application. Respective conditions are obtained for gyrostabilization systems in the critical case of zero roots. Equations of the theory of precession are taken as simplified equations.

1. We consider systems whose perturbed motion are defined by differential equations reduced to the form

$$\mu^2 \frac{dx_1}{dt} = P_1(\mu)x + X_1(t,\mu,z,x),\quad \mu\frac{dx_2}{dt} = P_2(\mu)x + X_2(t,\mu,z,x) \tag{1.1}$$

$$\frac{dx_3}{dt} = P_3(\mu)x + X_3(t,\mu,z,x),\quad \frac{dz}{dt} = Z(t,\mu,z,x)$$

where x_1, x_2, x_3, z are n_1-, n_2-, n_3-, m-dimensional vectors, $x = \|x_1, x_2, x_3\|^T$, T denotes transposition, μ is a small positive parameter, $P_i(\mu)$ $(i = 1,2,3)$ are matrices of corresponding dimensions whose elements are continuous functions of μ, Z, X_i $(i = 1,2,3)$ are vector functions holomorphic (in some domain) in the totality of variables x, z, which do not contain in their expansions terms of power lower than the second, whose coefficients are continuous bounded functions of t and μ. Let $Z(t,\mu,z,0) = 0$, $X_i(t,\mu,z,0) = 0$ $(i = 1,2,3)$.

Let us take as the simplified system that which is obtained from system (1.1) by retaining in its equations only terms that contain μ of power not exceeding the first

$$0 = P_1{}^*x + X_1{}^*,\quad \mu\frac{dx_2}{dt} = P_2{}^*x + X_2{}^* \tag{1.2}$$

$$\frac{dx_3}{dt} = P_3{}^*x + X_3{}^*,\quad \frac{dz}{dt} = Z^*$$

where the asterisk denotes terms retained in (1.1).

The characteristic equation of the first approximation of system (1.1) has m zero roots. The remaining roots are obtained from the equation

$$D(\lambda,\mu) = \begin{vmatrix} \mu^2\lambda E - P_{11} & -P_{12} & -P_{13} \\ -P_{21} & \mu\lambda E - P_{22} & -P_{23} \\ -P_{31} & -P_{32} & \lambda E - P_{33} \end{vmatrix} = 0 \tag{1.3}$$

where P_{ij} is a submatrix of matrix P_i of dimension $n_i \times n_j$ $(i, j = 1,2,3)$, with $D(\lambda,\mu)$ represented form

$$D(\lambda,\mu) = f_1(\lambda,\mu) + \mu^2 f_2(\lambda,\mu)$$

where $f_1(\lambda,\mu)$ is a polynomial in λ which is obtained from $D(\lambda,\mu)$, when in each element of the determinant only terms that contain μ in power not higher than the first are taken into account.

The characteristic equation for system (1.2) is of the form $\lambda^m D^*(\lambda,\mu) = 0$, where $D^*(\lambda,\mu) = f_1(\lambda,\mu)$. We shall call equation

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$$D^* (\lambda, \mu) = 0 \tag{1.4}$$

the shortened equation.

System (1.2) and its corresponding shortened equation are of a lower order than the complete system (1.1) and Eq.(1.3). Let us investigate the conditions under which stability of the zero solution of the simplified system (1.2) ensures the stability of zero solution of the complete system. A problem of this type was considered in /1/ in the case when the degenerate system obtained from (1.1) with $\mu = 0$ was used as the simplified system. The problem formulated here is of independent interest in applications.

To solve this problem we use the methods applied in /1/. We denote by λ and λ_* the roots of Eqs.(1.3) and (1.4), respectively. From the proof given in /1/ follows that:

when $D(0, 0) \neq 0$, n_3 roots λ and λ_* approach λ_0 roots of the degenerate equation $D(\lambda, 0) = 0$ as $\mu \to 0$, and at the limit are equal to them;

n_2 roots of these equations can be represented in the form $\lambda = \alpha(\mu)/\mu$ and $\lambda_* = \alpha_*(\mu)/\mu$, respectively, where $\alpha(\mu)$ and $\alpha_*(\mu)$ approach values of α_0 roots of the equation

$$d(\alpha) = \begin{vmatrix} P_{11}(0) & P_{12}(0) \\ -P_{21}(0) & \alpha E - P_{22}(0) \end{vmatrix} = 0 \tag{1.5}$$

when $|P_{11}(0)| \neq 0$ and $\mu \to 0$, and at the limit are equal to them. Evaluation of the error $\Delta\lambda = \lambda - \lambda_0$, $\Delta\lambda_* = \lambda_* - \lambda_0$ for roots of the first group and of $\Delta\alpha = \alpha - \alpha_0$, $\Delta\alpha_* = \alpha_* - \alpha_0$ for the second group enables us to show that for fairly small values of parameter μ $(n_2 + n_3)$ the roots of equation $D(\lambda, \mu) = 0$ have negative real parts, if the shortened equation satisfies the Hurwitz conditions. The remaining n_1 roots of Eq.(1.3) with fairly small μ have negative real parts when the equation

$$|\beta E - P_{11}(0)| = 0 \tag{1.6}$$

satisfies the Hurwitz conditions.

Taking this into account and reasoning as in /1/ enables us by applying the Liapunov theorem /2/ to show that the following theorem is valid.

Theorem 1. If for $D(0, 0) \neq 0$ Eq.(1.6) satisfies the Hurwitz conditions, and the roots of the characteristic equation of the simplified system have negative real parts (except the m zero roots), then for fairly small μ the stability of the zero solution of the simplified system (1.2) implies stability of the zero solution of system (1.1).

Note that in that case solution $z = C$, $x = 0$, where $\| C \|$ is fairly small, is also stable.

2. The theorem in Sect.1 enables us to use for solving the stability problem the system of simplified differential equations of a lower order than the complete system. The method used for prooving that theorem makes possible the evaluation of the upper limit of admissible values of parameter μ for which the above statement is valid.

From the viewpoint of application the problem of establishing the closeness of solutions of such systems is interesting. Suppose that the respective input data that define the solution of the complete (1.1) and simplified (1.2) systems are close to each other (or coincide). Under what conditions these solutions remain close over an infinite time interval? Problems of this type (on the admissibility to use a simplified system obtained by that or other method) were considered by many authors (e.g., /3—6/).

We denote by $x = x(t, \mu)$, $z = z(t, \mu)$ the solution of system (1.1) with initial conditions $x(t_0, \mu) = x_0$, $z(t_0, \mu) = z_0$; with $x^* = x^*(t, \mu)$, $z^* = z^*(t, \mu)$ representing the solution of the simplified system (1.2) defined by the initial conditions $x_2^*(t_0, \mu) = x_{20}^*$, $x_3^*(t_0, \mu) = x_{30}^*$, $z^*(t_0, \mu) = z_0^*$. Here $x_1^* = f_1(t, \mu, z^*, x_2^*, x_3^*)$, where $x_1 = f_1(t, \mu, z, x_2, x_3)$ is the solution of the algebraic equation in system (1.2)

$$0 = P_{11}^* x_1 + P_{12}^* x_2 + P_{13}^* x_3 + X_1^* \tag{2.1}$$

Theorem 2. If for $D(0, 0) \neq 0$ Eqs.(1.4) and (1.6) satisfy the Hurwitz conditions, then for the specified in advance numbers $\varepsilon > 0$ and $\delta > 0$ there exists a μ_* such that when $0 < \mu < \mu_*$ and $t > t_* > t_0$ we have in the perturbed motion

$$\| x - x^* \| < \varepsilon, \quad \| z - z^* \| < \varepsilon$$

if

$$x_2(t_0, \mu) = x_2^*(t_0, \mu), \quad x_3(t_0, \mu) = x_3^*(t_0, \mu)$$

$$z(t_0, \mu) = z^*(t_0, \mu), \quad \| x_1(t_0, \mu) - x_1^*(t_0, \mu) \| < \delta$$

By suitable selection of the small μ, t_* can be made as close as desired to t_0.

Proof. Consider the simplified system (1.2). Equation (2.1) under the stated above conditions and fairly small μ admits in conformity with the theorem on implicit function the unique solution $x_1 = f_1(t, \mu, z, x_2, x_3)$ in the form of a holomorphic function of variables x_2, x_3 with coefficients dependent on t, μ, z, and vanishing when $x_2 = 0, x_3 = 0$. Substituting this solution into the differential equations of system (1.2), we obtain

$$\mu \frac{dx_2}{dt} = P_2' \begin{Vmatrix} x_2 \\ x_3 \end{Vmatrix} + X_2', \quad \frac{dx_3}{dt} = P_3' \begin{Vmatrix} x_2 \\ x_3 \end{Vmatrix} + X_3', \quad \frac{dz}{dt} = Z' \tag{2.2}$$

where the nonlinear terms X_2', X_3', Z' vanish at $x_2 = 0, x_3 = 0$. Under these conditions the zero solution of system (2.2) is stable, as implied by the respective Liapunov theorem /2/. In perturbed motion we then have $x_2^* \to 0, x_3^* \to 0, z^* \to C^*$ and C^* is an arbitrary constant m-dimensional vector defined by the input data. Moreover, system (1.1) is of the type of Liapunov systems /2/. It follows from Sect.1 that with fairly small μ all conditions of the Liapunov theorem are satisfied for it, and in the perturbed motion $x \to 0, z \to C$.

Note that the integral

$$z + F(t, \mu, z, x) = A \tag{2.3}$$

occurs in the case of the complete system, and the integral

$$z + \varphi(t, \mu, z, x_2, x_3) = B \tag{2.4}$$

holds for the simplified system. In these integrals F and φ are holomorphic vector functions that vanish at $x = 0$, and $x_2 = 0, x_3 = 0$, respectively. Their expansions do not contain terms lower than the second power in z, x, and A and B are arbitrary constant vectors.

Consider solutions of systems (1.1) and (2.2) with fairly small μ, setting $x_{20} = x_{20}^*$, $x_{30} = x_{30}^*$, $z_0 = z_0^*$. We introduce variables $a = z - z^*$, $b_i = x_i - x_i^*$ $(i = 1, 2, 3)$, and shall consider the differential equations in variables b_i that correspond to noncritical variables which we obtain using Eqs.(1.1) and (2.2) and their integrals

$$\mu^2 \frac{db_1}{dt} = B_1(t, \mu, b_1, b_2, b_3), \quad \mu \frac{db_2}{dt} = B_2(t, \mu, b_1, b_2, b_3) \tag{2.5}$$

$$db_3/dt = B_3(t, \mu, b_1, b_2, b_3)$$

System (2.5) has no trivial solutions, and its right-hand sides vanish when $\mu = 0$, $b_i = 0$ $(i = 1, 2, 3)$.

Let us investigate the behavior of variables a, b over an infinite time interval. In conformity with the statement of the problem $a(t_0) = 0$, $b_j(t_0) = 0$ $(j = 2, 3)$, and $\| b_1(t_0) \| < \delta$, $\delta > 0$ is a number specified in advance. The analysis of system (2.5) and of the structure of integrals (2.3) and (2.4) will show that under the indicated conditions the derived solutions have the following property. In the case of specified in advance numbers ε and δ there exists a number $\mu_* > 0$ such that for all $t > t_* > t_0$ the inequalities $\| a \| < \varepsilon$, $\| b \| < \varepsilon$ are satisfied when the input data satisfy the adduced relations and when $\mu < \mu_*$. The theorem is proved.

Let us evaluate t_*. For this we substitute in the right-hand side of the differential equation for the variable b_1 of system (2.5) the solution $b_i = b_i(t, \mu)$ $(i = 1, 2, 3)$. Integrating the obtained identity in which $B_1(t, \mu, b_1(t, \mu), b_2(t, \mu), b_3(t, \mu))$ is a continuous function of t in limits from t_0 to t_*, we obtain the relation

$$\mu^2 (b_{1*} - b_{10}) = B_1' (t_* - t_0)$$

where B_1' is a vector with components

$$B_1^j(t_{1j}) \ (j = 1, \ldots, n_1); \ t_0 < t_{1j} < t_*, \quad b_{1*} = b_1(t_*), \ b_{10} = b_1(t_0); \ \| b_{1*} \| \leqslant \varepsilon, \ \| b_{10} \| < \delta.$$

By virtue of continuity for $t \in [t_0, t_*]$ we have $m_1 \leqslant | B_1^j | \leqslant M_1$ $(j = 1, \ldots, n_1)$, where m_1 and M_1 are positive constants.

Selecting $\xi > 0$ $\mu < m_1 \xi / (\varepsilon + \delta)$ as the specified in advance number, we obtain the estimate: $(t_* - t_0) < \xi$.

Remarks 1°. Using the equations of system (2.5) and integrals (2.3) and (2.4) it is possible to show that for fairly small μ relations $\| a \| < \varepsilon$, $\| b_2 \| < \varepsilon$, $\| b_3 \| < \varepsilon$ hold for all t commencing with t_0.

2°. It can be shown that the theorem remains generally valid, even when the input data that determine the solutions of the complete and simplified systems are not the same. In such case the following statement is valid. If $D(0,0) \neq 0$ and Eqs.(1.4) and (1.6) satisfy the Hurwitz conditions, then for the specified in advance numbers ε and δ there exist μ_* and η such that in perturbed motion with $0 < \mu < \mu_*$ and $t > t_* > t_0$

$$\| x - x^* \| < \varepsilon, \ \| z - z^* \| < \varepsilon$$

if

$$\|x_1(t_0,\mu)-x_1^*(t_0,\mu)\|<\delta,\quad \|x_2(t_0,\mu)-x_2^*(t_0,\mu)\|<\eta$$
$$\|x_3(t_0,\mu)-x_3^*(t_0,\mu)\|<\eta,\quad \|z(t_0,\mu)-z^*(t_0,\mu)\|<\eta$$

By a suitable selection of the small μ, t_* can be made as close to t_0 as desired.

3. As an example of application of obtained results, we consider gyroscopic systems containing gyroscopes of high proper moments of momentum. Since the differential equations of motion of such systems are complex, therefore their analysis is carried out in practice using approximate methods which must be rigorously substantiated. Particular difficulties arise in the analysis of systems of gyroscopic stabilization, such as those considered below.

We consider systems such as electromechanical ones on fixed bases that contain gyroscopes, taking into account some of the actual properties of their elements, as in /7/. For our model the differential equations of perturbed motion are of the form

$$\frac{d}{dt}a\dot{q}_M+(b^\circ+g^\circ)q_M'=Q_M'+Q_M'' \tag{3.1}$$
$$\frac{d}{dt}L\dot{q}_E+B^\circ\dot{q}_3+R^\circ\dot{q}_E=Q_E'+Q_E'',\quad \frac{dq_M}{dt}=\dot{q}_M$$
$$Q_M'=\begin{Vmatrix}0\\ A^\circ\dot{q}_E\\ -c^\circ q_4\end{Vmatrix},\qquad Q_E'=-\begin{Vmatrix}\omega^\circ q_1\\ \Omega^\circ\dot{q}_E\\ 0\end{Vmatrix}$$

(the notation of /7/ is retained, but without parametric perturbations). In these equations q_M is the n-dimensional vector of generalized mechanical coordinates, $q_i\ (i=1,\ldots,4)$ are vectors whose components are, respectively, the gyroscope precession angles, angles of deviation of the gyroscope proper rotation from their values is stable motion, the angles of turn of rotors of stabilizing motors, and deformation of elastic elements /7/; q_E is the u-dimensional vector of generalized electrical coordinates.

All functions in (3.1) are assumed to be holomorphic in totality of variables in some domain, with $Q_M''(q_M,\dot{q}_M,\dot{q}_E)$, $Q_E''(q_M,\dot{q}_M,\dot{q}_E)$ nonlinear vector functions that do not contain terms of power lower than the second. The small circle superscript denotes terms of zero order in expansions of respective functions.

Let us consider systems that contain gyroscopes with large proper moments of momentum, and assume that $g=g^*H$, where H is a large positive dimensionless parameter. We denote the small parameter by $\mu=H^{-1}$.

We introduce, as in /1/, the new time $\tau=\mu t$ and carry out the substitution of variables

$$z=\mu^2a_1\frac{dq_M}{d\tau}+(\mu b_1^\circ+g_1^{*\circ})q_M,\quad x_1=a\frac{dq_M}{d\tau}$$
$$x_2=L\dot{q}_E,\ x_3=q_1,\ x_4=q_4$$

where a_1, b_1, g_1 are submatrices of dimension $m\times n$ of matrices a, b, g, respectively. In the new variables system (3.1) is of the form

$$\mu^2\frac{dx_1}{d\tau}=-(\mu b'+g')x_1+X_M'+X_M'' \tag{3.2}$$
$$\mu\frac{dx_2}{d\tau}=-\mu B'x_1-R'x_2+X_E'+X_E''$$
$$dx_3/d\tau=d_1x_1,\ dx_4/d\tau=d_4x_1,\ dz/d\tau=Z$$

where the prime denotes transformed matrices, X_M' and X_E' are functions Q_M' and Q_E' in terms of respective new variables, and Z, X_M'', X_E'' are holomorphic vector functions whose expansions do not contain terms of power lower than the second, and

$$Z(x_1=0)=0,\ X_M''(x=0)=0,\ X_E''(x=0)=0;\ x=\|x_1,x_2,x_3,x_4\|^T$$

System (3.2) is of the form of system (1.1). As in Sect.1, we consider as approximate the system which we obtain from (3.2), retaining in equations of the latter only terms that contain μ of power not higher than the first. We denote this approximate system by $(3.2)_*$, without presenting it in explicit form.

In the old variables the corresponding system is

$$(b^\circ+g^\circ)\frac{dq_M}{dt}=Q_M'+\bar{Q}_M'' \tag{3.3}$$
$$\frac{d}{dt}L\dot{q}_E+B^\circ\dot{q}_3+R^\circ\dot{q}_E=Q_E'+Q_E''$$

which we take as the simplified one. Note that Eqs. (3.3) coincide with the precession equations used in the applied theory of gyroscopes. These equations appear, for instance, in /8 — 10/.

Applying the results of Sects. 1 and 2, and taking into account the singularities of gyroscopic systems indicated in /1/, it is possible to show that when $|g^\circ| \neq 0$, $|g_{kj}^\circ|_{k=1,\ldots,m}^{j=l+1,\ldots,s} \neq 0$, in spite of m zero roots, the remaining roots of the characteristic equation of the first approximation system for $(3.2)_*$ lie in the left-hand half-plane, and the equation $|\beta E + \mu b' + g'| = 0$ satisfies the Hurwitz conditions, then for fairly small μ stability of the zero solution of the approximate system $(3.2)_*$ ensures, also, the stability of the zero solution of the complete system (3.2).

Reverting to old variables and taking into account that the nonsingular uniformly regular transformation of variables retains stability properties, we obtain for the considered here gyroscopic systems the following theorem (the asterisk denotes the solution of system (3.3), and $q_M^{\cdot *} = f_M(q_M^*, q_E^{\cdot *})$, where f_M is the solution of equation $(b^\circ + g^\circ) q_M^{\cdot} = Q_M' + \overline{Q}_M''$ for $q_M^{\cdot}$):

Theorem 3. If for $|g^\circ| \neq 0$, $|g_{kj}^\circ|_{k=1,\ldots,m}^{j=l+1,\ldots,s} \neq 0$ all roots, except the m zero ones, of the characteristic equations of the simplified system (3.3) have negative real parts, and the equation

$$|a^\circ\lambda + b^\circ + g^\circ| = 0 \tag{3.4}$$

satisfies the Hurwitz conditions, then for fairly large parameter H the stability of the zero solution of the simplified system (3.3) implies the stability of zero solution of the complete system (3.1), and for any numbers $\varepsilon > 0$ and $\delta > 0$ and all $H > H_*$ the inequalities

$$\|q_M^{\cdot} - q_M^{\cdot *}\| < \varepsilon, \quad \|q_M - q_M^*\| < \varepsilon, \quad \|q_E^{\cdot} - q_E^{\cdot *}\| < \varepsilon$$

are satisfied in the case of perturbed motion for $t > t_* > t_0$, if

$$\|q_{M0}^{\cdot} - q_{M0}^{\cdot *}\| < \delta, \quad q_{M0} = q_{M0}^*, \quad q_{E0}^{\cdot} = q_{E0}^{\cdot *}$$

By an appropriate selection of large H, t_* can be made as close to t_0 as desired.

The theorem was obtained for the particular case when input conditions in terms of generalized mechanical coordinates and electrical generalized velocities for the complete and the simplified systems are the same. In conformity with remarks in Sect. 2 the theorem can be extended to the case of different input data. Then for all $t \in [t_0, \infty)$ we have $\|q_E^{\cdot} - q_E^{\cdot *}\| < \varepsilon$, $\|q_M - q_M^*\| < \varepsilon$.

The described here investigations complement the results previously obtained in /8,11, 12/.

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